

Dimensional and statistical foundations for accumulated damage models

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Abstract This paper develops a framework for creating damage accumulation models for engineered wood products by invoking the classical theory of non-dimensionalization. The result is a general class of such models. Both the US and Canadian damage accumulation models are revisited. It is shown how the former may be generalized within that framework while deficiencies are discovered in the latter and overcome. Use of modern Bayesian statistical methods for estimating the parameters in these models is proposed along with an illustrative application of these methods to a ramp load dataset.

Introduction

The reliability of manufactured lumber products used in structural engineering applications is assured by their design values. These values would be relatively easy to specify for short-term loadings, for example, in terms of estimated fifth percentiles of their breaking strengths under such loads. But these design values must also account for the combination of short- and long-term future dead and live loads they must sustain. For this purpose, the theory of accumulated damage models (ADMs) was developed (for a review see, e.g., Hoffmeyer and Sørensen 2007; Zhai 2011).

The seminal paper of Wood (1951) played a key role in that development, by showing that the strength of lumber is a relative thing—it depends on how loads are applied. His empirical assessments showed conclusively that the load at failure will

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be much higher when that load is applied at a rapidly increasing rate as compared with a slowly increasing rate. This is called the duration-of-load effect and Wood quantified it in the so-called Madison curve (Cai 2015). But these empirical assessments did not enable the strength of lumber under future loadings to be characterized. For that purpose, models were needed.

ADMs were developed to meet that need. These models are parametric functions over time of the future stress loading profile $\tau(t), t \geq 0$; the damage accumulated over a long future could then be predicted using a $\tau(t)$ that reflected the types of loads, for example, snow, that might occur. Their ingenuity derived from the feasibility of estimating the model parameters from data obtained from accelerated testing. In other words, in laboratory experiments $\tau(t)$ could be chosen to ensure failure in a realistic time frame, for example, a ramp load test of duration about 1 min to yield the short-term strength τ_s of a piece of lumber. That short-term strength τ_s could be treated as a property of a piece of lumber randomly selected from any given in-grade population of interest. Or it could, depending on the context, be treated as a fixed parameter of that population. But in either case, its role could be expressed very simply in ADMs through the unitless stress ratio $\sigma(t) = \tau(t)/\tau_s$ —the impact on a random piece of lumber of a general load profile at time t would be calibrated by the multiplicative rescaling factor τ_s^{-1} . That stress ratio became a fundamental determinant in models of the rate at which damage to a piece of lumber accumulated over time as the future load is applied.

A general form for an ADM is given in Rosowsky and Bulleit (2002), which states that damage accumulates at a rate determined by

$$\frac{d\alpha(t)}{dt} = g(\alpha(t), \sigma(t), \mathbf{v}), \quad (1)$$

where $\alpha(t)$ is the damage accumulated up to time t , g is a function to be specified, \mathbf{v} is a vector of parameters, and $\sigma(t)$ is the applied stress ratio at time t as defined above. The general model in Eq. (1) has been extended by Köhler and Svensson (2002) to include additional model parameters ξ to be fitted using experimental data, where the form of the model is derived from engineering theory, for example, crack formation theory. While the focus here is models based on Eq. (1), the approaches discussed are also applicable to the more general models.

The accumulated damage $0 \leq \alpha(t) \leq 1$ is a non-decreasing function of t . At time 0, $\alpha(0) = 0$, as no damage will yet have occurred. It is assumed that the loading is sufficiently large as to cause failure at a finite random time $t = T_1$, at which time accumulation of damage is complete and that α is scaled so that $\alpha(T_1) = 1$. Note that the “accumulated damage” $\alpha(t)$ is latent—it is not an observable characteristic of the piece of lumber. Instead it provides a framework on which to hang the various elements of the model.

An important feature of the models is their temporal scales. The values of τ_s , $\sigma(t)$, and $\tau(t)$ at any fixed point in time do not depend on the units of measurement chosen for t (e.g., whether time is measured in minutes or seconds). But the same cannot be said of the rate at which damage is accumulated as specified by Eq. (1). This rate must by definition depend on the unit scale adopted for the dimension of

time. In fact the dimensions and the scales on which they are measured are a fundamental aspect of any general theory for a natural phenomenon. However, the ADMs have been inconsistent in the way time and other parameters have been incorporated in the model, leading to unrecognized technical anomalies (Cai 2015). Thus, the first major result of this paper is to develop a new approach to constructing ADMs by non-dimensionalizing the problem.

Fitting ADMs has proven challenging, because Eq. 1 does not readily yield a likelihood function, which is the cornerstone of conventional statistical approaches for estimating model parameters. Instead various complicated methods for estimating those parameters were developed (Foschi and Yao 1986; Gerhards and Link 1987), although statistical properties such as their standard errors are difficult to assess. Thus, a second major contribution of this paper is a new and principled statistical foundation based on the use of Bayesian methods to incorporate both randomness between specimens and model error; while much more computationally intensive, the abundance of modern computing power makes their applicability to these models now feasible. This new approach necessitated the development of code for implementation over a large cluster of CPU cores, used in this paper to compare two well-known ADMs after appropriate non-dimensionalization.

To summarize, “New models through dimensional analysis” section shows in detail how one may develop a model for a natural process, in this case damage accumulation, by first non-dimensionalizing to canonical form, thus bypassing the need for scales of measurement. Then, variations of models are developed that have been of fundamental importance in the development of design values that account for uncertainties in future loading profiles. “The experiment” section describes the data obtained from a ramp load experiment in the FPInnovations testing laboratory. “Data analysis” section provides a novel illustrative application where models are fitted and compared using the Bayesian statistical methods described in this paper. The paper concludes with a brief discussion in “Conclusion” section.

New models through dimensional analysis

This section focuses on the concept of dimension as it relates to ADMs. The following is an illustrative example.

Example 1 Suppose two scientists S_{\min} and S_{hr} working, respectively, on timescales of minutes m and hours h are engaged on a damage modeling project. Their respective objectives are accumulated damage models, α_{\min} and α_{hr} . They know the stress ratios, $\sigma_{\min}(m)$ and $\sigma_{\text{hr}}(h)$, respectively, and also that $\sigma_{\text{hr}}(h) = \sigma_{\min}(60h)$ since they are working on the same project. They also know in the end that their times to failure must be the same $M = 60H$, that $\alpha_{\text{hr}}(h) = \alpha_{\min}(60h)$, $h \geq 0$ and that $1 = \alpha_{\text{hr}}(H) = \alpha_{\min}(60H)$, $h \geq 0$.

They now proceed to solve Eq. (1) to get

$$\begin{aligned}\alpha_{\min}(m) &= \int_0^m g(\alpha(m'), \sigma(m'), \mathbf{v}) dm' \\ \alpha_{\text{hr}}(h) &= \int_0^h g(\alpha(h'), \sigma(h'), \mathbf{v}) dh'.\end{aligned}\quad (2)$$

To check her results, S_{hr} does further analysis and finds

$$\begin{aligned}\alpha_{\text{hr}}(h) &= \int_0^h g(\alpha_{\text{hr}}(h'), \sigma_{\text{hr}}(h'), \mathbf{v}) dh' \\ &= \int_0^h g(\alpha_{\min}(60h'), \sigma_{\min}(60h'), \mathbf{v}) dh' .\end{aligned}$$

Changing the variable of integration, $m' = 60h'$, yields

$$\alpha_{\text{hr}}(h) = \frac{\int_0^{60h} g(\alpha_{\text{hr}}(m'), \sigma_{\text{hr}}(m'), \mathbf{v}) dm'}{60} = \frac{\alpha_{\min}(60h)}{60},$$

leading to a contradiction since $1 = \alpha_{\text{hr}}(H) = \alpha_{\min}(60H)/60 = \alpha_{\min}(M)/60 = 1/60$.

This contradiction in Example 1 would be resolved if a standard unit was adopted for measuring time. Then, a time duration may be expressed as $\{t\}[t]$, where $[t]$ denotes that standard unit and $\{t\}$ denotes the number of those units. That approach has been used to define the index pH of acidity of an aqueous solution: it is defined by $pH = -\log_{10}\{a_H^+\}$ where $\{a_H^+\}$ is the number of internationally agreed on units of its hydrogen ion concentration. Thus, pH becomes unitless and $pH = 7$ for example, always represents the acidity of distilled water. This approach also bypasses another problem that as a transcendental function, the logarithm $x \rightarrow \log(x)$ cannot be applied to x unless it is unitless (Matta et al. 2010).

However, this technical “fix” does not seem satisfactory for modeling the strength properties of lumber. Different time units may be preferable to others in certain contexts, in particular since ADMs are used in both short-term testing and long-term reliability. More importantly, if $\{T_1\}$ were the random time to failure on the standardized scale, results could not be interpreted on another timescale, e.g., “hours.” As the above analysis shows, different results would be obtained if the model had been built say on the hourly timescale, and now everything including the fitted model parameters were to be synchronized to that standard unit of time.

An alternative solution would include a time rescaling factor. The following example illustrates the approach using a special case of model (1).

Example 2 As originally formulated the US accumulated damage model (Gerhards and Link 1987) is given by

$$\dot{\alpha}(t) = \exp\{-A + B\sigma(t)\} \quad (3)$$

where the “dot” means derivative with respect to time, and the parameter vector here is $\mathbf{v} = (A, B)$. This model cannot be correct as formulated, since the

exponential function $x \rightarrow \exp(x)$ cannot be applied to x unless the latter is unitless. This can be corrected by defining

$$\dot{\alpha}(t) = C([t]) \exp\{-A + B\sigma(t)\}$$

where $C([t])$ depends on t only through the units $[t]^{-1}$ on the timescale on which it is measured, i.e., the units of $C([t])$ are the inverse of the units of t . Then, model (3) yields in this special case

$$\begin{aligned} \alpha_{\text{hr}}(h) &= \int_0^h C([h']) \exp\{-A + B\sigma_{\text{hr}}(h')\} dh' \\ &= \int_0^h C([h']) \exp\{-A + B\sigma_{\min}(60h')\} dh'. \end{aligned}$$

If the variables are transformed as above $m' = 60h'$, $C([h']) \rightarrow C([m'])60$ so that the 60 cancels out in Eq. (2), thereby eliminating the inconsistency seen above.

However, the approach illustrated in Example 2 proves impractical in cases that have many parameters with associated units of measurement. Of greater concern then is the possibility that the model itself is dimensionally inconsistent, in which case it could not be said to represent a natural phenomenon (Shen 2015). These considerations lead to the approach taken in this paper, of reducing the model to its canonical form by non-dimensionalizing it and hence eliminating the scale altogether and thus concerning about the units in which things are measured. The approach, developed in the next subsection, shows one well-known ADM to have inconsistencies, that cannot be simply resolved as suggested above by including a scale parameter.

Non-dimensionalizing models

Although dimensional analysis has a long history (Bluman and Kumei 2013), this paper focuses on the celebrated Buckingham π theorem (Buckingham 1914), which resolves the inconsistencies noted above. That theorem assumes that the scientist has specified a meaningful and complete set of quantities (or variables) Q_1, \dots, Q_n for the phenomenon under investigation. The goal is a model that specifies their relationship:

$$f(Q_1, \dots, Q_n) = 0. \quad (4)$$

The remarkable π theorem shows that under mild conditions, this characterizing relationship can always be re-expressed in a simpler, dimensionless form through what Buckingham calls π functions, which satisfy $F(\pi_1, \dots, \pi_p) = 0$, where the $\pi_i = \prod_{j=1}^n Q_j^{a_{ij}}$ are dimensionless and hence unitless. The $\{\pi\}$'s can thus be considered to be the fundamental building blocks of the relationship expressed in Eq. (4). Moreover, estimating F may be much simpler than estimating f based on experimental data, since p can sometimes be much smaller than n .

This section shows how the theorem can be applied, specifically to develop alternatives to well-known ADMs that ensure dimensional consistency. But the method can be applied more generally in developing engineering models with the potential benefit of simplifying the experiments needed to fit the core relationship among the quantities related to the models. One famous example, from fluid dynamics, concerns the force F on a body immersed in a fluid stream, which depends on the body's length L , fluid velocity V , fluid density ρ , and fluid viscosity μ . Of interest is the relationship g

$$F = g(L, V, \rho, \mu).$$

An experiment designed to estimate g would be complex since all five of these quantities would seemingly need to vary. However the approach to be described below, when applied to this case shows that the fundamental relationship among these quantities actually involves just two quantities, one being the dimensionless Reynolds number for the fluid, $Re = \rho VL/\mu$. More precisely

$$F^* = g^*(Re),$$

where $F^* = F/(\rho L^2 V^2)$. A much simpler experiment yields an estimate g^* from which the desired estimate of g can be found.

The application in this paper has as a primary goal, to relate the rate at which the damage accumulation model $\alpha(t)$ changes at a time t , to other features of a randomly chosen specimen of an engineered wood product. Denote that change by $\dot{\alpha}(t) = \frac{\partial \alpha(t)}{\partial t}$. Through its random quantities, that model represents the population from which that specimen is drawn. For those quantities, the models that have already been proposed, such as those seen in the sequel, guided the selection of specific versions of Eq. (4).

In general, the dimensions of quantities are represented by using square bracket notation. Thus, a quantity Q would be written as $Q = \{Q\}[Q]$ where $[Q]$ is the dimension of Q while $\{Q\}$ is the number of units the quantity has in that dimension. The primary (or basic) dimensions of interest in this application are time, mass, and length, denoted by T , M , and L , respectively. Note that length would be used for any dimension of size including width, height, and thickness. Thus, a quantity with dimension time would be written as $t = \{t\}[t]$ where $[t] = T$. Once a dimension like time has been identified, a scale has to be assigned according to how that dimension is to be quantified or measured.

A key element of an ADM is its rate of change, $Q_1 = \dot{\alpha}(t)$ with $[Q_1] = T^{-1}$. Modelers (see, for example, Foschi and Yao 1986) have assumed that it depends in a Markovian way on the accumulated damage, i.e., on $Q_2 = \alpha(t) \in [0, 1]$, $[Q_2] = 1^0$, a dimensionless quantity. The value $\alpha = 1$ is reached when the random specimen fails. Note that $\dot{\alpha}(t)$ may depend on t only indirectly, through some other quantity. The rate of change at a specific time also depends on the stress $Q_3 = \tau(t)$, $[Q_3] = FA^{-1}$ where $A = L^2$ denotes the dimension of area and F denotes the dimension of force. As noted above, the short-term breaking strength $Q_4 = \tau_s$, $[Q_4] = FA^{-1}$, plays a key role. It has generally been represented by the breaking strength under a ramp

load test of short duration with a loading profile $\tau(t) = kt$ for a constant load rate k , $[k] = FT^{-1}$, that is

$$\tau_s = kT_s, \quad (5)$$

where $T_s, [T_s] = T$ is the short-term breaking time. As applied to modeling the accumulated damage, T_s is a latent characteristic of a piece of lumber or its corresponding population parameter, whereas k would be known.

Note that Eq. (5) also holds approximately under another type of short-term test where it is the deflection rate, not the load rate, which is held constant. For completeness, that type of ramp test will now be described along with the k involved, which must now depend on the piece of lumber (Conroy Lum, personal communication). To illustrate the calculation of k in a simple case, suppose the piece is anchored at its ends in a bending machine. The span or distance between the supports is L^* . Two downward acting loads $F^*/2$ are each applied at equi-spaced points along the span. In reaction, this induces upward acting loads $F^*/2$ at the ends of the span for a total of four loads acting on the member. Standard beam theory implies that at time t the maximum deflection at mid span is

$$D(\text{inches}) = \frac{F^*L^*/3}{48EI} \left[3L^{*2} - 4\left(\frac{L^*}{3}\right)^2 \right] = \frac{23F^*L^{*3}}{1296EI} \quad (6)$$

where $E(\text{psi})$ is the specimen-specific measure of elasticity, and $I = bd^3/12$, the areal moment of inertia, b being the breadth of the member and d being its depth. Equation (6) may thus be simplified as

$$D(\text{inches}) = \frac{276F^*L^{*3}}{1296Ebd^3}.$$

During the test, the force will dynamically increase over time so that at time t

$$D(t)(\text{inches}) = \frac{276F^*(t)L^{*3}}{1296Ebd^3} = \frac{CF^*(t)}{E},$$

for a constant $C > 0$. Thus, requiring a constant deflection rate $\dot{D}(t) = d$ implies

$$d = C\dot{F}^*(t)/E, \quad (7)$$

which means $F^*(t) = kt$ where

$$k = Ed/C. \quad (8)$$

Equation (7) shows that if a constant deflection rate d is to be maintained over time, the force $F^*(t)$ must be adjusted to a higher value when E is large than when it is small. In general, the calculation above would need to be adapted to the particular test being used. But it does show that k can be calculated explicitly, knowing E , so is not a random effect. It also shows that the effect of E is absorbed in k so it needs not

be included as a quantity in the model. Thus, in τ_s only the time to failure T_s (with $[T_s] = T$) is random.

For more general testing scenarios that differ from the standard ramp load, it is not T_s (with $[T_s] = T$) that is observed, but rather the time to failure under a given load profile $\tau(t)$ which shall be denoted by T_1 . When the accumulation of damage is complete, the specimen fails and $\alpha(T_1) = 1$. While failure time T_1 is clearly an important quantity, it is specimen-specific and derived from α and hence needs not explicitly be included in the model. Instead, a reference level for time that is estimable from the experimental data might be used, for example, the population average time of T_1 denoted by μ_1 . That feature is therefore included in the model as $Q_5 = \mu_1, [Q_5] = T$.

The rate of change in $\tau(t)$, $Q_6 = \dot{\tau}(t), [Q_6] = F(AT)^{-1}$, has not been considered in previous models. But for completeness, it is now shown how it could be made part of the general framework.

Finally, there is the size of a specimen as a determinant of the rate at which damage is accumulated. Size would be characterized by a number of features, depending on the nature of the product. For definiteness, assume just three, $Q_7 = Width$, $Q_8 = Thickness$ and $Q_9 = Length$. Formally, they all have the dimension of length L . Thus, for example, $[Thickness] = L$. These features will all be constants when interest focuses on a specific size class. But in the context of modeling the full in-grade population based on a random sample, these quantities will vary and thus are included in the general model as well.

The above list of quantities with their units is summarized in Table 1.

The π theorem can be applied in various ways, depending on which dimensions are chosen as the primary ones, and which the secondary. Note that in the summary above only three primary or reference dimensions, L , F , and T are manifest. This implies there are just 3 so-called repeating quantities and $9 - 3 = 6$ π functions. The repeating quantities cannot include the model’s predictand Q_1 . Previous work has shown Q_4 to be important as a baseline measure of strength. The average failure time of the population seems a good choice given its importance as a parameter. Q_9 could well be chosen to represent the length L group of quantities, especially if the population specimens were of fixed length, but of varying width and thickness.

These considerations suggest forming the $\{\pi\}$ functions by first eliminating Q_4 as well as Q_9 and then successively modifying Q_1, Q_2, Q_3, Q_6, Q_7 , and Q_8 . To illustrate the process, Q_1 is added to the repeating variables Q_4^b and Q_8 to form the first π function as

Table 1 Q functions for accumulated damage models

	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8	Q_9
Quantity	$\dot{\alpha}(t)$	$\alpha(t)$	$\tau(t)$	τ_s	μ_1	$\dot{\tau}(t)$	W	T	L
Units	T^{-1}	1^0	FA^{-1}	FA^{-1}	T	FA^{-1}	L	L	L

Here *Width*, *Thickness* and *Length* are abbreviated by W , T and L for brevity

$$\pi_1 \doteq Q_1 Q_4^a Q_5^b Q_9^c$$

with a , b , and c chosen to make π_1 dimensionless. This is interpreted in dimensional terms as

$$(T^{-1})(FA^{-1})^a (T)^b (L)^c = F^0 T^0 L^0$$

giving $a = 0$, $b = 1$ and $c = 0$. Thus

$$\pi_1 = Q_1 = \alpha(\dot{t})\mu_1.$$

Similarly,

$$\pi_2 \doteq Q_2 Q_4^a Q_5^b Q_9^c, \text{ and hence, the restriction} \\ (1^0)(FL^{-2})^a (T)^b (L)^c = F^0 T^0 L^0$$

meaning that

$$\pi_2 = Q_2 = \alpha(t);$$

Continuing,

$$\pi_3 \doteq Q_3 Q_4^a Q_5^b Q_9^c \\ (FL^{-2})(FL^{-2})^a (T)^b (L)^c = F^0 T^0 L^0,$$

which yields

$$\pi_3 = Q_3 Q_4^{-1} = \frac{\tau(t)}{\tau_s}.$$

The remaining π functions can be obtained in a similar fashion:

$$\pi_6 \doteq Q_6 Q_4^{-1} Q_5 = \frac{\dot{v}(t)\mu_1}{\tau_s} \\ \pi_7 \doteq Q_6 Q_9^{-1} = \frac{\text{Width}}{\text{Length}} \\ \pi_8 \doteq Q_7 Q_9^{-1} = \frac{\text{Thickness}}{\text{Length}}.$$

Buckingham's theorem implies $F(\pi_1, \dots, \pi_p) = 0$, or

$$\pi_1 = F^*(\pi_2, \pi_3, \pi_6, \pi_7, \pi_8), \quad (9)$$

that is

$$\dot{\alpha}(t)\mu_1 = F^* \left(\alpha(t), \frac{\tau(t)}{\tau_s}, \frac{\dot{\tau}(t)\mu_1}{\tau_s}, \frac{Width}{Length}, \frac{Thickness}{Length} \right). \quad (10)$$

Remarks:

1. This application of Buckingham's theory eliminates length as predictive of the rate of accumulative damage in agreement with the standard models like those in "The US model" and "The Canadian model" sections. But those models unlike the ones proposed in this paper also exclude width and length. This may be reasonable in the case of short-term (ramp) tests since the cross-sectional area is already represented in the moment of areal inertia, that in turn, like the modulus of rupture, is absorbed in coefficient k in Eq. (7). But the rationale for this exclusion for an arbitrary loading curve $\tau(t)$ is unclear to these authors.
2. Although Eq. (10) was developed with reference to a specific time point t , the same relationship holds for all $t \in [0, T_1]$ where T_1 denotes the time at which the specimen fails. Hence, the π functions that are expressed as functions of t are genuinely time-dependent.
3. Equation (10) provides a fundamental relationship among all the quantities in the characterizing relationship given in Eq. (4). The functions f and F remain to be specified by some combination of scientific methods and experimental work. As they are models for a randomly selected specimen, they will be random. Moreover, they, like all models, will be inexact and hence require the inclusion of an uncertain model error; the Bayesian context of the paper requires they must be treated as random. Examples of ways of incorporating that uncertainty follow in the sequel.

The US model

This section introduces a special case of the model in Eq. (9), namely the one in Eq. (3) for which

$$\begin{aligned} \pi_1(t) &= F^*(\pi_3(t)) \\ &= \exp \left\{ -A + B \frac{\tau(t)}{\tau_s} \right\}. \end{aligned} \quad (11)$$

Here, A and B are random effects that reflect residual model uncertainty since τ_s does not capture all the variation from specimen-to-specimen. This model is an amended version of the so-called US model in that now unlike before, the left-hand side is dimensionless in agreement with the right.

Integration yields

$$\alpha(t)\mu_1 = \exp\{-A\} \int_0^t \exp \left\{ B \frac{\tau(t')}{\tau_s} \right\} dt'. \quad (12)$$

Specifically at the failure time $t = T_1$,

$$\mu_1 = \exp\{-A\} \int_0^{T_1} \exp\left\{B \frac{\tau(t')}{\tau_s}\right\} dt'. \tag{13}$$

Observe that Eq. (11) implies $\pi_1(0) = \dot{\alpha}(0)\mu_1 = \exp\{-A\}$, so using this in Eq. (13) gives

$$\dot{\alpha}(0) = \left[\int_0^{T_1} \exp\left\{B \frac{\tau(t')}{\tau_s}\right\} dt' \right]^{-1}.$$

Then dividing by μ_1 , (12) can be re-expressed as

$$\begin{aligned} \alpha(t) &= \dot{\alpha}(0) \int_0^t \exp\left\{B \frac{\tau(t')}{\tau_s}\right\} dt' \\ &= \frac{\int_0^t \exp\left\{B \frac{\tau(t')}{\tau_s}\right\} dt'}{\int_0^{T_1} \exp\left\{B \frac{\tau(t')}{\tau_s}\right\} dt'} \\ &= \frac{\int_0^{(t/\mu_1)} \exp\left\{B \frac{\tau(u\mu_1)}{\tau_s}\right\} du}{\int_0^{(T_1/\mu_1)} \exp\left\{B \frac{\tau(u\mu_1)}{\tau_s}\right\} du} \end{aligned} \tag{14}$$

by the change of variables $u = t'/\mu_1$, which is exactly what was obtained in “Non-dimensionalizing models” section by the *ad hoc* approach taken there.

For the special case of a ramp load test the substitutions $T_1 = T_s$, $\mu_1 = \mu_s$, and $\tau(t) = kt$ can be made, where k is the known loading rate (which may vary between specimens if a constant deflection rate is maintained), $[k] = FL^{-2}T^{-1}$, $\tau_s = kT_s$, and μ_s is the average short-term strength. Integrating Eq. (13) directly gives the failure time T_s in terms of A , B , and μ_s as

$$T_s = \frac{\mu_s \cdot B \exp\{A\}}{\exp\{B\} - 1} \tag{15}$$

and Eq. (14) implies

$$\begin{aligned} \alpha(t) &= \frac{\int_0^{(t/\mu_s)} \exp\left\{\frac{Bu}{T_s/\mu_s}\right\} du}{\int_0^{(T_s/\mu_s)} \exp\left\{\frac{Bu}{T_s/\mu_s}\right\} du} \\ &= \frac{\exp\{Bt/T_s\} - 1}{\exp\{B\} - 1}. \end{aligned}$$

Remarks:

4. Observe that for the ramp load test

$$\begin{aligned} \dot{\alpha}(0) &= [\mu_s \exp\{A\}]^{-1} \\ \dot{\alpha}(T_s) &= \frac{B \exp\{B\}}{T_s \exp\{B\} - 1}. \end{aligned}$$

These equations provide some intuition on the role of the random effects A and B . The first equation shows that the initial rate of damage accumulation for a specimen relative to the population is governed by its random effect A ; for a small A that rate is faster. The second equation shows that B controls the accumulation rate as the specimen approaches its failure time; for large B that rate is faster, since $\frac{B \exp\{B\}}{\exp\{B\}-1}$ is an increasing function of B .

The Canadian model

This subsection provides another instance of the model in Eq. (9) in what is now referred to as the “Canadian model” (Foschi and Yao 1986). As originally specified it is given by

$$\dot{\alpha}(t) = a[\tau(t) - \sigma_0\tau_s]_+^b + c[\tau(t) - \sigma_0\tau_s]_+^n \alpha(t), \tag{16}$$

where a, b, c, n, σ_0 are log-normally distributed random effects, τ_s (psi) is the short-term breaking strength, $\tau(t)$ (psi) is the applied stress at time t , σ_0 is the stress ratio threshold, and $x_+ = \max(x, 0)$ so that the two terms in square brackets are non-negative. As before, the conditions $\alpha(0) = 0$ and $\alpha(T_1) = 1$ will determine T_1 for the given $\tau(t)$ as a function of the specimen-specific random effects.

As with the US model, the first step non-dimensionalizes time, by replacing the left-hand side of Eq. (16) by $\pi_1(t)$. Then the right-hand side must be unitless as well; however, as formulated the units associated with both terms on the right-hand side of the model involve powers, b and n . These lead, respectively, to units in those terms of $(psi)^b$ and $(psi)^n$. But the coefficients, a and c , cannot involve those random powers and so cannot compensate to make those two terms unitless. A simple adjustment re-expresses that equation using $\pi_3(t) = \tau(t)/\tau_s$, so that

$$\begin{aligned} \pi_1(t) &= [(\tilde{a}\tau_s)(\tau(t)/\tau_s - \sigma_0)_+]^b + [(\tilde{c}\tau_s)(\tau(t)/\tau_s - \sigma_0)_+]^n \alpha(t) \\ &= [(\tilde{a}\tau_s)(\pi_3(t) - \sigma_0)_+]^b + [(\tilde{c}\tau_s)(\pi_3(t) - \sigma_0)_+]^n \alpha(t), \end{aligned}$$

where \tilde{a} and \tilde{c} are now random effects with $[\tilde{a}] = [\tilde{c}] = F^{-1}L^2$.

To illustrate the use of this model, again consider the special case of a ramp load test where $T_1 = T_s$, $\tau_s = kT_s$, and $\pi_3(t) = t/T_s$. Then

$$\dot{\alpha}(t)\mu_s = [\tilde{a}kT_s(t/T_s - \sigma_0)_+]^b + [\tilde{c}kT_s(t/T_s - \sigma_0)_+]^n \alpha(t). \tag{17}$$

As before, the loading rate k is known and may be specimen specific. Define the integrating factor

$$\begin{aligned} H(t) &= \exp\left\{ \int -\frac{1}{\mu_s} \left[\tilde{c}kT_s \left(\frac{t}{T_s} - \sigma_0 \right) \right]^n dt \right\} \\ &= \exp\left\{ -\frac{1}{\mu_s} (\tilde{c}kT_s)^n \frac{T_s}{n+1} \left(\frac{t}{T_s} - \sigma_0 \right)^{n+1} \right\}. \end{aligned}$$

Then

$$\frac{d}{dt}[\alpha(t)H(t)] = \frac{1}{\mu_s} \cdot H(t) \left[\tilde{a}kT_s \left(\frac{t}{T_s} - \sigma_0 \right) \right]^b.$$

For this model, no damage is accumulated until the stress ratio threshold reaches $t = \sigma_0 T_s$. Integration then yields

$$\alpha(T_s)H(T_s) - \alpha(\sigma_0 T_s)H(\sigma_0 T_s) = \int_{\sigma_0 T_s}^{T_s} \frac{1}{\mu_s} \cdot H(t) \left[\tilde{a}kT_s \left(\frac{t}{T_s} - \sigma_0 \right) \right]^b dt.$$

The change in variables $u = -\log H(t)$ and the positivity of $-\log H(t)$ yields

$$H(T_s) = \frac{(\tilde{a}kT_s)^b}{(\tilde{c}kT_s)^{n(b+1)/(n+1)}} \left(\frac{\mu_s(n+1)}{T_s} \right)^{\frac{b-n}{n+1}} \int_0^{-\log H(T_s)} e^{-u} u^{(b+1)/(n+1)-1} du,$$

the integral being the lower incomplete Gamma function, which can be evaluated numerically using standard mathematical libraries. Given the values of random effects a , b , c , n , and σ_0 , T_s are determined by the solution to this equation. Unlike the US model however, this equation does not have an analytical solution and must be solved numerically.

Remarks:

- Some later authors (e.g., Köhler and Svensson 2002; Hoffmeyer and Sørensen 2007) state the Canadian model in the following manner instead:

$$\dot{\alpha}(t) = a \left(\frac{\tau(t)}{\tau_s} - \sigma_0 \right)_+^b + c \left(\frac{\tau(t)}{\tau_s} - \sigma_0 \right)_+^n \alpha(t). \quad (18)$$

Note that this is an alternative way to resolve the inconsistent *psi* units on the right-hand side of Eq. (16). However, it also fundamentally changes the nature of the dependence of $\dot{\alpha}(t)$ on τ_s . In particular, fitting the model to ramp load data, using the specification in Eq. (18) will not explicitly depend on the loading rate k , while the original Canadian model does.

The experiment

The experimental data consist of $n = 98$ specimens of 12-ft 1650f-1.5E Spruce-Pine-Fir (SPF) 2x4 randomly drawn from a bundle and tested destructively under short-term bending loads. The bending machine was set up for a span corresponding to a span-to-depth ratio of 21:1 (73.5 inches) for testing in accordance with ASTM D 4761, Section 6–10 (ASTM 2005). The edge to be stressed in tension was selected randomly, and the maximum strength reducing characteristic was randomly located in the 73.5-inch test span. Specimens were trimmed to remove the excess overhang, allowing for 4 inches past each end (a total length 81.5 inches).

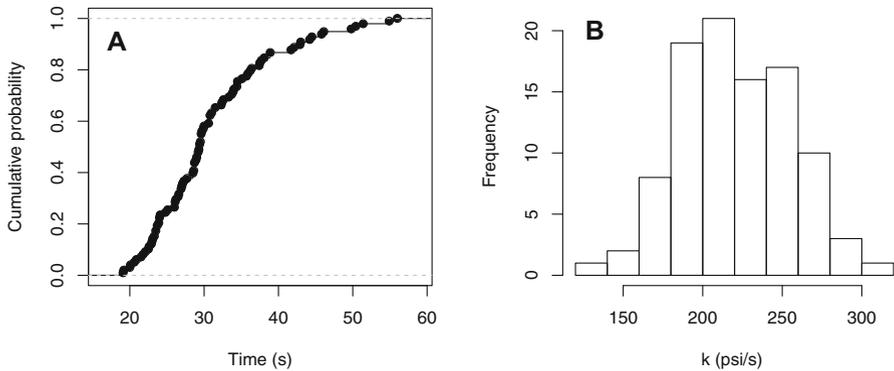


Fig. 1 **a** Empirical cumulative distribution of short-term bending failure times; **b** ramp loading rate of specimens

The load profile was set to a constant deflection rate of $d = 0.045$ in/s. This deflection rate translates to approximate ramp load tests with a specimen-specific loading rate k that depends on its elasticity E . As shown in Eq. (8), the loading rate k is approximately linear in E , and thus the variability in k can be attributed to the variability in E among the specimens in the sample. The time until failure (T_s) was recorded for each specimen. Figure 1 shows the empirical cumulative distribution of T_s and a histogram of the realized loading rates k for the sample. Based on these data, set the reference failure time μ_s to be the sample mean of 31.0 s.

Data analysis

Overview of Bayesian statistical methods

Methods to fit the experimental data to the models discussed in “The US model” and “The Canadian model” sections are now developed. For the US model, assume that A and B in Eq. (15) are specimen-specific random effects, log-normally distributed with parameters (μ_A, σ_A^2) and (μ_B, σ_B^2) , respectively. For the Canadian model, assume \tilde{a} , b , \tilde{c} , n , and σ_0 in Eq. (17) are specimen-specific random effects. Assume as in the original Canadian model’s derivation, that the unitless random effects b and n are log-normally distributed, with respective parameters (μ_b, σ_b^2) , (μ_n, σ_n^2) . Since the stress ratio satisfies $0 < \sigma_0 < 1$, the normal distribution may be adopted for σ_0 after a logit-transformation, with parameters $(\mu_{s0}, \sigma_{s0}^2)$. The remaining random effects \tilde{a} and \tilde{c} are problematical since they have units $F^{-1}L^2$, thus ruling out use of the log-normal distribution as was done in the past. Instead the normal distribution has been chosen as an approximation, with parameters (μ_a, σ_a^2) and (μ_c, σ_c^2) , respectively, so that now these parameters can have the appropriate units. In all cases, μ is the mean parameter and σ^2 refers to the variance parameter of the distribution. Finally, while the theoretical failure times T_s are deterministic

solutions to equations involving the random effects, this condition may be relaxed to accommodate model error.

A Bayesian statistical approach is adopted for fitting these models (Gelman et al. 2014). As far as the authors know, such methods have not been previously applied to estimate parameters for ADMs, so a brief review is provided along with a description of their merit in the problem at hand. Bayesian analysis combines two ingredients: the “prior,” which is a probability density specified on the parameters to represent the investigator’s knowledge before the experiment is done; and the likelihood function of the observed data given parameter values. The latter is the basis of the classical “maximum likelihood” approach to parameter estimation. In the Bayesian setting, the product of the prior and likelihood gives the “posterior” distribution, which represents the probability distribution of the parameters after seeing the data. That posterior is the basis of drawing conclusions about the parameters.

The advantages of a Bayesian approach for estimating ADMs are twofold. First, uncertainty about the parameters is captured. This is of particular importance since the random effects are not observed in the data; only the failure time T_s and loading rate k . Therefore, it is difficult to obtain reliable confidence intervals on the parameters by matching theoretical and empirical quantiles as was done in the past, see for example, Foschi and Yao (1986). This is handled naturally in the Bayesian setting, since the posterior can be explored effectively using Markov Chain Monte Carlo (MCMC) techniques to provide genuine posterior probability intervals. Second, the posterior can be used to construct predictive distributions for model checking or prediction for future specimens. When the posterior is explored via MCMC simulation, such predictive distributions can be easily obtained by using the MCMC samples to numerically integrate out unknowns from the fitted model, as will be demonstrated.

Analysis procedure

In this subsection, the procedure to carry out the Bayesian analysis on the dataset and compare models is described.

The likelihood functions for both the US and Canadian models are first needed. Let θ denote the vector of model parameters, and T_i, γ_i denote the failure time and vector of unobserved random effects for specimen i , respectively. In general, let $h(\gamma)$ denote the deterministically solved failure time corresponding to a random effect vector γ ; since this solution does not readily yield a tractable likelihood function, an approximation is adopted by assuming that these solutions have accuracy to the nearest second for data on the current timescale (~ 30 s). This assumption accommodates model error and implies that the observed T_s lies uniformly randomly in the interval $[h(\gamma) - 0.5s, h(\gamma) + 0.5s]$. Recall that the US solution is given analytically in Eq. (15), while the Canadian solution must be found numerically.

The general notation $p(a|b)$ is used to denote the probability distribution of a conditional on b . Then, the likelihood L of the parameters θ for specimen i is

$$\begin{aligned}
 L(\theta|T_i) &= p(T_i|\theta) \\
 &= \int p(T_i|\gamma_i)p(\gamma_i|\theta) d\gamma_i \\
 &= \int I[h(\gamma_i) - 0.5 \leq T_i \leq h(\gamma_i) + 0.5] \times p(\gamma_i|\theta) d\gamma_i,
 \end{aligned}$$

where I denotes the indicator function where $I(x) = 1$ if x is true, and 0 otherwise.

This integral cannot be done analytically, but can be evaluated using Monte Carlo integration: draw N realizations of γ_i from its distribution given the current values of θ , where N is a large integer. Denote these values by $\gamma_i^{(1)}, \gamma_i^{(2)}, \dots, \gamma_i^{(N)}$. Then, a large enough N yields a result arbitrarily close to the true likelihood value via the estimate

$$L(\theta|T_i) \approx \frac{1}{N} \sum_{j=1}^N I \left[h(\gamma_i^{(j)}) - 0.5 \leq T_i \leq h(\gamma_i^{(j)}) + 0.5 \right], \quad (19)$$

which in other words is simply the proportion of samples where the observed T_i is compatible with γ . Note that this likelihood does not have an analytical gradient, and the necessity of Monte Carlo integration in its calculation would render numerical gradients to be unstable. Hence, a direct maximization of the likelihood function (for a maximum likelihood analysis) is not straightforward, but this poses no difficulty for the MCMC techniques adopted here.

Next, priors must be specified on the parameters. Here it is assumed, *a priori*, that all the parameters in θ are statistically independent. Let the μ parameters have a $Normal(0, 100^2)$ prior density and the σ^2 parameters have a $Inv\text{-Gamma}(0.001, 0.001)$ prior density. These choices of priors represent the absence of any prior knowledge on the parameters (Gelman 2006). Then, assuming the test sample consists of n statistically independent specimens, the posterior distribution of the parameters is given by

$$p(\theta|T_1, \dots, T_n) \propto p(\theta) \prod_{i=1}^n p(T_i|\theta),$$

where $p(\theta)$ denotes the joint probability density of the priors.

When the posterior is analytically intractable, as is the case here, inference can be made by drawing representative samples from this probability distribution using MCMC simulation techniques (Brooks et al. 2011). The particular variant of MCMC used here for efficiency is parallel tempering (Swendsen and Wang 1986) on the power posterior with Metropolis–Hastings iterations on each computing node. Empirical assessments suggest that using $N = 10,000$ draws in Eq. (19) provides sufficiently reliable calculations (absolute error in the log-posterior < 1.0).

In the Bayesian setting, model comparison is often carried out by calculating the Bayes factor, to determine which model is more strongly supported by the data (Kass and Raftery 1995). The Bayes factor in favor of the Canadian model ($M = 1$) versus the US model ($M = 2$) is defined as:

$$B_{12} = \frac{p(T_1, \dots, T_n | M = 1)}{p(T_1, \dots, T_n | M = 2)}, \tag{20}$$

where the term $p(T_1, \dots, T_n | M = m)$ is known as the *marginal likelihood* of model m . The calculation of the marginal likelihood integrates out the model parameters, thus taking into account the model complexity and number of parameters. Hence, the Bayes factor, which is the ratio of the marginal likelihoods, directly evaluates which of the two models is more strongly supported by the data, with $B_{12} > 1$ indicating that model $M = 1$ is more strongly supported by the data than $M = 2$; $B_{12} > 150$ is generally considered as “very strong” or decisive evidence (Kass and Raftery 1995). Here to calculate the numerical value of Eq. (20) from the MCMC samples, Equation (7) in Friel and Pettitt (2008) was used.

Finally, suppose an application of interest is to use model m to predict the failure time T_f for a specimen. The Bayesian framework provides the probability distribution of T_f as

$$p(T_f | M = m) = \int p(T_f | \theta, M = m) p(\theta | T_1, \dots, T_n, M = m) d\theta \tag{21}$$

This distribution can be applied to predict failure times of future specimens and to check the quality of the model fit on the existing data. These are illustrated in the following section.

Table 2 Summaries of posterior distributions of parameters for the US and Canadian models

	Posterior quantiles		
	50%	2.5%	97.5%
<i>US model</i>			
μ_A	0.643	−0.479	1.439
σ_A	0.100	0.020	0.356
μ_B	1.15	0.25	1.81
σ_B	0.036	0.015	0.119
Marginal log-likelihood: −339.7			
<i>Canadian model</i>			
μ_a	1.97	0.228	4.36
σ_a	0.0357	0.0162	0.33
μ_b	1.84	−1.56	4.13
σ_b	0.0741	0.0153	0.72
μ_c	2.29	0.252	6.29
σ_c	0.0317	0.0147	0.551
μ_n	−1.33	−5.96	1.98
σ_n	0.0521	0.0162	0.931
μ_{s0}	1.58	−2.69	2.79
σ_{s0}	0.0435	0.0158	0.307
Marginal log-likelihood: −326.27			

Results

The fitted US and Canadian models based on the experimental data are presented first. Each computing node in the parallel tempering MCMC setup ran 10,000 Metropolis–Hastings iterations, with the first 1000 samples discarded as burn-in. Table 2 summarizes the key quantiles from the resulting posterior distributions of the parameters. Consider the 50% quantile (median) to be the point estimate of each parameter; the 2.5 and 97.5% quantiles can be interpreted as the endpoints of the 95% Bayesian credible interval, i.e., the posterior probability that the parameter lies within it is 0.95. The marginal likelihoods of the two models (on the log-scale) are also shown; these yield the Bayes factor $B_{12} = 6.8 \times 10^5$, and this magnitude of B_{12} suggests the data strongly favors the Canadian model (Kass and Raftery 1995) for this particular dataset.

To assess how each model fits, Eq. (21) was used to generate 100 hypothetical replicates of the dataset. A visual of the fit quality is obtained by superimposing the empirical cumulative distributions of the replicates (in gray), onto the actual cumulative distribution of the data shown in Fig. 1. The results are shown in Fig. 2. Notice that while the central portions of the distributions appear to fit equally well, the Canadian model is better able to replicate the observed data in both the lower and upper extremes of the distribution. There is less variability in the replicates (gray) around the observed data for the specimens with the shorter and longer failure times.

Finally Fig. 3 depicts two plots of predictive distributions based on the fitted Canadian model for this sample, again computed using Eq. (21). For this purpose, two different ramp loading rates are compared: the slower rate $k = 0.1$, and the faster rate $k = 0.3$. These predictive distributions corroborate the expected effect: the group subject to the faster loading rate sustains a higher average load at failure. The mean time to failure of the $k = 0.3$ scenario is 23.8 s, compared to 59.6 s for

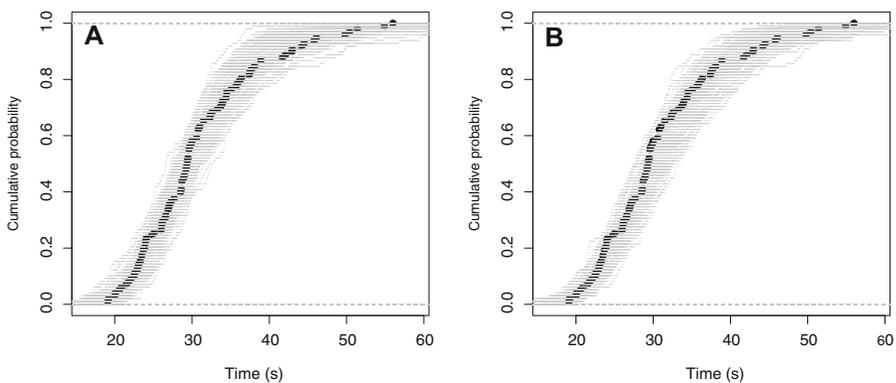


Fig. 2 Assessment of model fits via empirical cumulative distributions of data generated from fitted models. **a** US model, **b** Canadian model. It can be seen that the intervals for the Canadian model are narrower in both the lower and upper extremes of the distribution

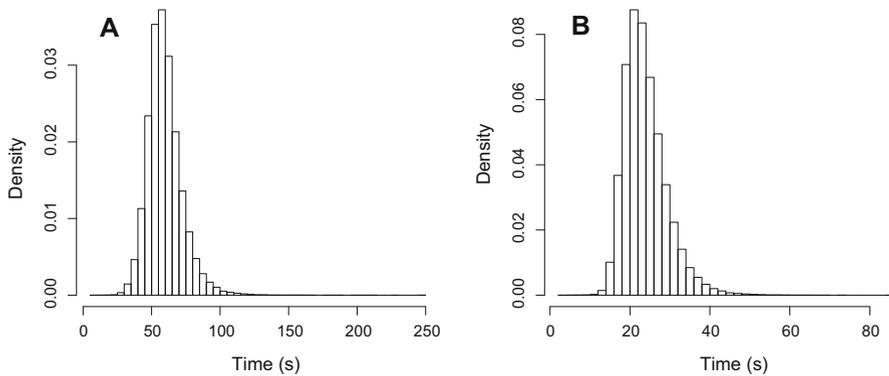


Fig. 3 Predictive distributions for the Canadian model using two different loading rates. **a** Loading rate $k = 0.1$, **b** loading rate $k = 0.3$. The average load sustained at failure is higher for the faster loading rate

the $k = 0.1$ scenario. These correspond to average loads at failure of 7158 *psi* and 5956 *psi*, respectively.

Conclusion

In this paper, a framework based on dimensional analysis was presented that enables one to build accumulated damage models. The analysis in “Non-dimensionalizing models” section shows the need to ensure dimensional coherence in model development. There two investigators working on different timescales but using the very same accumulated damage model can reach strikingly different conclusions about the rate of damage accumulation. The problem for that model can be solved merely by changing it to recognize that as a transcendental function, $\exp(x)$, neither the function nor its argument x can have units of measurement. But a deeper analysis, based primarily on an application of the celebrated Buckingham π theorem (Buckingham 1914), ensures that the model does not depend on what scales are used for measurement. The final result is a family of possible accumulated damage models from which to select a model in a specific application.

The paper then explores how two well-known models—the US model and the Canadian model—can be adapted to fit into that family while retaining their important features. The *ad hoc* approach in “Non-dimensionalizing models” section yielded conclusions about the US model. But for the Canadian model, substantially more adaptation was needed.

The second major feature of this paper was a demonstration of how the resulting models could then be implemented within a Bayesian statistical framework in order to reflect all their associated uncertainties. That demonstration was carried out in the simplest case of ramp load testing using experimental data that the first author produced in an FPInnovations Vancouver laboratory. The empirical results for that dataset favored the Canadian model. In this case, only one timescale for loading was considered, i.e., an average failure time of ~ 30 s, to show the merits of the

Bayesian approach for working with these models. The same statistical approach applies for analyzing data from different timescales. Classic studies on rate-of-loading (e.g., Karacabeyli and Barrett (1993)) have used ramp load tests with different loading rates (e.g., with average failure times set to 5 h, 10 min, 1 min, 1 s, etc.), as well as constant load tests, to quantify the effect of load rate on strength. That more extensive analysis will be the subject of follow-up work to this paper.

Overall, the paper has provided a foundation for accumulated damage modeling on which can be used to build new models for setting design values for new engineered lumber products such as cross laminated timber or strand-based wood composites (see, for example, Wang et al. 2012a, 2012b).

One might well ask whether such a foundation is needed. After all, the original Canadian model did fit the experimental data rather well despite its dimensional inconsistencies—that is, when implicitly the units of measurement were dropped. The good fit is perhaps not surprising given the large number of parameters in the model. One is reminded of John von Neumann’s famous quip: “With four parameters I can fit an elephant and with five I can make him wiggle his trunk.”

The authors’ response would be that such models, which can only be fitted on accelerated test data, cannot be directly validated for their intended use in predicting long-term reliability. Therefore, they must be developed in accordance with good modeling practice, to ensure that they appear trustworthy. In particular the period of time until a piece of lumber fails does not depend on the units in which that period is measured, as ensured by application of the Buckingham π theorem in this paper. Another important feature of good modeling practice embraced in this paper is a method for fitting the model that comes with a characterization of the uncertainties associated with it.

Finally, unpublished work by the authors done since the current paper was first submitted, based on differences in the way the analysis could be done as well as in the models, shows important differences in the results given by the analyses of the original Canadian model and the non-dimensionalized version presented here (Yang et al. 2017).

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